VARIATION OF THE MODULUS OF A FOLIATION

M. CISKA

ABSTRACT. The p–modulus $\mathrm{mod}_p(\mathcal{F})$ of a foliation $\mathcal{F}$ on a Riemannian manifold M is a generalization of extremal length of plane curves introduced by L. Ahlfors. We study the variation $t \mapsto \mathrm{mod}_p(\mathcal{F}_t)$ of the modulus. In particular, we consider product of moduli of orthogonal foliations.

1. Introduction

Modulus, an inverse of extremal length, of a family of a plane curves is a conformal invariant [1]. The notion of a modulus can be generalized to any family of submanifolds [4, 2] and, hence, to a foliation. Roughly speaking, the p–modulus $\mathrm{mod}_p(\mathcal{F})$ of a k–dimensional foliation on a n–dimensional Riemannian manifold is the infimum of p–th norm over all nonnegative, p–integrable functions f such that $\int_L f \geq 1$ for almost every $L \in \mathcal{F}$. If $n = kp$, then the amount $\mathrm{mod}_p(\mathcal{F})$ is a conformal invariant.

In this paper, we study the variation of a modulus. We generalize the result obtained by Kalina and Pierzchalski [5] for codimension one foliations given by a submersion. We assume the existence of a function f_0 which realizes the p–modulus and do not put any requirements on the dimension and codimension of a foliation on a Riemannian manifold (M, g). The methods used here are different than the one used in [5] and rely on a integral formula obtained by the author in [3]:

$$\int_M f_0^{p-1}\varphi \, d\mu_M = \int_M f_0^p \widehat{\varphi} \, d\mu_M, \quad \text{where} \quad \widehat{\varphi}(x) = \int_{L_x} \varphi \, d\mu_{L_x}.$$

The main formula is the following

$$\frac{d}{dt}\mathrm{mod}_p(\mathcal{F}_t)^p_{t=0} = -p \int_M f_0^{p-1} \left(g(\nabla f_0, X) + f_0 \mathrm{div}_{\mathcal{F}_0} \mathrm{X} \right) d\mu_M. \tag{1.1}$$

which is valid for all *admissible* foliations i.e. foliations satisfying certain assumptions (see Theorem 4.1). We show that all foliations given by a submersion are admissible (Theorem 3.1).

Using the formula (1.1) we obtain conditions for a foliation to be a critical point of a variation. We show that foliation is a critical point of a variation if and only if the gradient of extremal function f_0 is a combination of

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mean curvature of a foliation and distribution orthogonal to this foliation (Corollary 4.4).

2. Preliminaries

Let (M,g) be a Riemannian manifold, $\mathcal{F}$ a k–dimensional foliation on M. Let μ_M and μ_L denote Lebesgue measures on M and $L \in \mathcal{F}$, respectively. Fix the coefficient $p > 1$ and let $L^p(M)$ be a space of all p–integrable functions on M with respect to μ_M with the norm $\|f\|_p = \left(\int_M |f|^p \, d\mu_M\right)^{\frac{1}{p}}$. Denote by $\mathrm{adm}_p(\mathcal{F})$ a subfamily of $L^p(M)$ of all nonnegative functions f such that $\int_L f \, d\mu_L \geq 1$ for almost every $L \in \mathcal{F}$. The *p–modulus* $\mathrm{mod}_p(\mathcal{F})$ of $\mathcal{F}$ is defined as follows

$$\mathrm{mod}_p(\mathcal{F}) = \inf_{f \in \mathrm{adm}_p(\mathcal{F})} \|f\|_p$$

if $\mathrm{adm}_p(\mathcal{F}) \neq \emptyset$ and $\mathrm{mod}_p(\mathcal{F}) = \infty$ otherwise [3]. Function $f_0 \in \mathrm{adm}_p(\mathcal{F})$ which realizes the modulus i.e.

$$\|f_0\|_p = \mathrm{mod}_p(\mathcal{F})$$

is called *extremal function* for p–modulus of $\mathcal{F}$. An extremal function does not exist for any foliation. Namely, we have the following characterization of existence of f_0.

Proposition 2.1 ([3])**.** *There exists an extremal function for p–modulus of a foliation $\mathcal{F}$ if and only if for any subfamily $\mathcal{L} \subset \mathcal{F}$ such that* $\mathrm{mod}_p(\mathcal{F}) = 0$ *we have $\mu(\bigcup \mathcal{L}) = 0$.*

Remark 2.2. Notice that the modulus for any subfamily $\mathcal{L} \subset \mathcal{F}$ is defined in the same way considering functions defined on M not only on $\bigcup \mathcal{L}$.

The extremal function has the following properties.

Proposition 2.3 ([3])**.** *Assume there exists an extremal function f_0 for p–modulus of $\mathcal{F}$. Then*

1. $\int_L f_0 = 1$ *for almost every leaf $L \in \mathcal{F}$,*
2. $f_0 > 0$,
3. *for any $\varphi \in L^p(M)$ we have $\varphi \in L^1(L)$ for almost every leaf $L \in \mathcal{F}$.*

Assume there exists an extremal function for a p–modulus of a foliation $\mathcal{F}$. Then, by Proposition 2.3, for any $\varphi \in L^p(M)$ we have $\varphi \in L^1(L)$ for almost every $L \in \mathcal{F}$. Hence, the following function

$$\widehat{\varphi}(x) = \int_{L_x} \varphi \, d\mu_{L_x}, \quad x \in L_x \in \mathcal{F}.$$

is well defined.

Theorem 2.4 ([3])**.** *Let f_0 be an extremal function for p–modulus of $\mathcal{F}$. Let $\varphi \in L^p(M)$ be such that* $\operatorname{esssup}|\varphi| < \infty$ *and* $\operatorname{esssup}|\widehat{\varphi}| < \infty$. *Then*

$$\int_M f_0^{p-1} \varphi \, d\mu_M = \int_M f_0^p \widehat{\varphi} \, d\mu_M.$$

Consider now a foliation $\mathcal{F}$ given by the level sets of a submersion $\Phi : M \to N$ i.e. $\mathcal{F} = \{\Phi^{-1}(y)\}_{y \in N}$. Decompose the tangent bundle TM into vertical and horizontal distributions

$$TM = \mathcal{V} \oplus \mathcal{H}, \quad \mathcal{V} = \ker \Phi_*, \quad \mathcal{H} = \mathcal{V}^\perp,$$

where $\perp$ denotes the orthogonal complement with respect to Riemannian metric on M. Then $\Phi_{*x} : \mathcal{H}_x \to T_{\Phi(x)}N$, $x \in M$, is a linear isomorphism. Let $\Phi_{*x}^* : T_{\Phi(x)}N \to \mathcal{H}_x$ be an adjoint linear operator. The Jacobian $J\Phi$ of Φ is equal

$$J\Phi(x) = \sqrt{\det(\Phi_{*x} \circ \Phi_{*x}^* : \mathcal{H}_x \to \mathcal{H}_x)}, \quad x \in M.$$

The condition for existence of an extremal function for a foliation given by the level sets of a submersion takes the following form.

Proposition 2.5 ([3])**.** *Let $\mathcal{F}$ be a foliation defined by a submersion $\Phi : M \to N$ such that $J\Phi < C$ for some constant C. Let L_x denotes the leaf of $\mathcal{F}$ through $x \in M$ and put $\mathcal{F}_\infty = \{x \in M : \mu_{L_x}(L_x) = \infty\}$. Assume moreover $\mu_M(M) < \infty$. Then, there is an extremal function for p–modulus of $\mathcal{F}$ (for any $p > 1$) if and only if $\mu_M(\mathcal{F}_\infty) = 0$.*

There is an explicit formula for an extremal function in the case of a foliation given by a submersion.

Proposition 2.6 ([5, 3])**.** *If f_0 is an extremal function for p–modulus of a foliation $\mathcal{F}$ given by the level sets of a submersion $\Phi : M \to N$, then*

$$f_0 = \frac{(J\Phi)^{\frac{1}{p-1}}}{\widehat{(J\Phi)^{\frac{1}{p-1}}}}.$$

3. Admissible foliations

Let (M, g) be a Riemannian manifold, $p > 1$. Let X be a compactly supported vector field on M, φ_t a flow of X. Let $\mathcal{F}$ be a foliation on M and put $\mathcal{F}_t = \varphi_t(\mathcal{F})$.

We say that X is *admissible* for p–modulus of $\mathcal{F}$ if, for some interval $I = (-\varepsilon, \varepsilon)$, we have

(A1) there exists an extremal function f_t for p–modulus of $\mathcal{F}_t$ for all $t \in I$,
(A2) the function $\alpha(x, t) = (f_t \circ \varphi_t)(x)$ is C^1–smooth with respect to variable $t \in I$,
(A3) there is $h_1 \in L^p(M)$ such that $|\alpha(x, t)| < h_1(x)$ for all $t \in I$,

(A4) there is $h_2 \in L^p(M)$ such that $|\frac{\partial\alpha}{\partial t}(x,t)| < h_2(x)$ for all $t \in I$.

In addition, if every compactly supported vector field is admissible for p–modulus of $\mathcal{F}$, then we say that $\mathcal{F}$ is *p–admissible.*

The main result of this section is the following.

Theorem 3.1. *Let $\mathcal{F}$ be a foliation on a Riemannian manifold M given by the level sets of a submersion $\Phi : M \to N$. Assume*

(1) *$C_1 < J\Phi < C_2$ and $\hat{1} < C_3$ for some positive constants C_1, C_2, C_3,*
(2) *an extremal function for p–modulus of $\mathcal{F}$ is smooth ($p > 1$).*

Then $\mathcal{F}$ is p–admissible.

Proof. Let X be compactly supported vector field on M and let φ_t be a flow of X. Put $\mathcal{F}_t = \varphi_t(\mathcal{F})$. Then $\mathcal{F}_t$ is given by the level sets of the submersion $\Phi_t = \Phi \circ \varphi_t^{-1} : M \to N$. Moreover

$$J\Phi_t = J\Phi \cdot J^{\perp}\varphi_t^{-1},$$

where $J^{\perp}\varphi_t^{-1}$ is a smooth function depending only on differential φ_{t*} of the map φ_t.

Let L_z^t denotes the leaf of $\mathcal{F}_t$ through $z \in M$ i.e.

$$L_z^t = \Phi_t^{-1}(\Phi_t(z)), \quad z \in M.$$

We divide the proof into few steps.

Step 1 – there exist an extremal function f_t for p–modulus of $\mathcal{F}_t$.
Since $C_1 < J\Phi < C_2$ and φ_t is a flow of compactly supported vector field then the Jacobian $J\Phi_t$ is bounded. Since $\mu_M(\mathcal{F}_\infty) = 0$, then $\mu_M((\mathcal{F}_t)_\infty) = 0$, hence, by Proposition 2.5, there exists an extremal function f_t for p–modulus of $\mathcal{F}_t$. By Proposition 2.6 we have

$$\begin{aligned} f_t(z) &= \frac{(J\Phi_t)^{\frac{1}{p-1}}(z)}{\int_{L_z^t} (J\Phi_t)^{\frac{1}{p-1}}\, d\mu_{L_z^t}} \\ &= \frac{(J\Phi \circ \varphi_t^{-1}(z))^{\frac{1}{p-1}} (J^{\perp}\varphi_t^{-1}(z))^{\frac{1}{p-1}}}{\int_{L^0_{\varphi_t^{-1}(z)}} (J\Phi)^{\frac{1}{p-1}} ((J^{\perp}\varphi_t^{-1}) \circ \varphi_t)^{\frac{1}{p-1}} J^{\top}\varphi_t \, d\mu_{L^0_{\varphi_t^{-1}(z)}}}. \end{aligned} \tag{3.1}$$

Step 2 – function $t \mapsto \alpha(x,t) = (f_t \circ \varphi_t)(x)$ is C^1–smooth.
By (3.1)

$$(f_t \circ \varphi_t)(x) = \frac{(J\Phi(x))^{\frac{1}{p-1}} (J^{\perp}\varphi_t^{-1}(\varphi_t(x)))^{\frac{1}{p-1}}}{\int_{L_x^0} (J\Phi)^{\frac{1}{p-1}} ((J^{\perp}\varphi_t^{-1}) \circ \varphi_t)^{\frac{1}{p-1}} J^{\top}\varphi_t \, d\mu_{L_x^0}} \tag{3.2}$$

Functions

$$t \mapsto \beta_1(x,t) = (J^{\perp}\varphi_t^{-1}(\varphi_t(x)))^{\frac{1}{p-1}}$$

and

$$t \mapsto \beta_2(x,t) = ((J^{\perp}\varphi_t^{-1}) \circ \varphi_t)^{\frac{1}{p-1}} J^{\top}\varphi_t$$

are smooth and positive. Since X is compactly supported, for any closed interval I containing $0 \in \mathbb{R}$, functions

$$\beta_1(x,t), \quad \frac{\partial \beta_1}{\partial t}(x,t), \quad \beta_2(x,t), \quad \frac{\partial \beta_2}{\partial t}(x,t), \quad (x,t) \in M \times I$$

are bounded. By Lebesgue dominated convergence theorem, function $I \ni t \mapsto \alpha(x,t)$ is differentiable. Analogously, we show that this function is twice differentiable, hence is C^1–smooth.

Step 3 – $|f_t \circ \varphi_t| < Cf_0$ and $|\frac{d}{dt}(f_t \circ \varphi_t)| < Cf_0$ for some $C > 0$. By (3.2)

$$|(f_t \circ \varphi_t)(x)| = \left| \frac{(J\Phi)^{\frac{1}{p-1}} \beta_1(x,t)}{\int_{L_0^x} (J\Phi)^{\frac{1}{p-1}} \beta_2(x,t)\, d\mu_{L_0^x}} \right| \leq C \frac{(J\Phi)^{\frac{1}{p-1}}}{\int_{L_x^0} (J\Phi)^{\frac{1}{p-1}} d\mu_{L_x^0}} = Cf_0$$

and

$$\begin{aligned} \left| \frac{d}{dt} (f_t \circ \phi_t) \right| &= \left| \frac{(J\Phi)^{\frac{1}{p-1}} \frac{\partial \beta_1}{\partial t}(x,t)}{\int_{L_x^0} (J\Phi)^{\frac{1}{p-1}} \beta_2(x,t)\, d\mu_{L_x^0}} \right. \\ &\qquad \left. - \frac{(J\Phi)^{\frac{1}{p-1}} \beta_1(x,t) \cdot \int_{L_x^0} (J\Phi)^{\frac{1}{p-1}} \frac{\partial \beta_2}{\partial t}(x,t)\, d\mu_{L_x^0}}{\left(\int_{L_x^0} (J\Phi)^{\frac{1}{p-1}} \beta_2(x,t)\, d\mu_{L_x^0} \right)^2} \right| \\ &\leq C' \left| \frac{(J\Phi)^{\frac{1}{p-1}}}{\int_{L_x^0} (J\Phi)^{\frac{1}{p-1}} d\mu_{L_x^0}} \right| + C'' \left| \frac{(J\Phi)^{\frac{1}{p-1}} \int_{L_x^0} (J\Phi)^{\frac{1}{p-1}} d\mu_{L_x^0}}{\left(\int_{L_x^0} (J\Phi)^{\frac{1}{p-1}} d\mu_{L_x^0} \right)^2} \right| \\ &= Cf_0 \end{aligned}$$

Step 4 – $\inf f_0 > 0$.
Follows from the fact that $C_1 < J\Phi < C_2$ and $\hat{1} < C_3$ i.e.

$$f_0 \geq \frac{C_1^{\frac{1}{p-1}}}{C_2^{\frac{1}{p-1}} C_3} > 0.$$

□

By above theorem we get immediately the following corollary.

Corollary 3.2. *Let $\mathcal{F}$ be a foliation given by the level sets of a submersion $\Phi : M \to N$, where M is compact. Assume an extremal function for p–modulus of $\mathcal{F}$ is smooth. Then $\mathcal{F}$ is p–admissible.*

4. Variation of modulus

In this section, we consider the variation of p–modulus under the flow of compactly supported vector field. The formula for a variation implies some results about an extremal function.

Let $\mathcal{F}$ be a k–dimensional foliation on a Riemannian manifold (M,g). Denote by $\mathrm{div}_{\mathcal{F}}X$ the divergence of a vector field $X \in \Gamma(TM)$ with respect to leaves of $\mathcal{F}$ i.e.

$$\mathrm{div}_{\mathcal{F}}X = \sum_{i=1}^{k} g(\nabla_{e_i}X, e_i),$$

where $e_1,\ldots,e_k$ is a local orthonormal basis of $T\mathcal{F}$ and ∇ the Levi–Civita connection on M. Let $H_{\mathcal{F}}$ and $H_{\mathcal{F}^\perp}$ denote the mean curvatures of $\mathcal{F}$ and the distribution $\mathcal{F}^\perp$ orthogonal to $\mathcal{F}$, respectively. If X is tangent to $\mathcal{F}$ then the divergence $\mathrm{div}_M X$ on M and the divergence on the leaves $\mathrm{div}_{\mathcal{F}}X$ are related as follows

$$\mathrm{div}_M X = \mathrm{div}_{\mathcal{F}}X - g(X, H_{\mathcal{F}^\perp}), \quad X \in \Gamma(T\mathcal{F}). \tag{4.1}$$

Moreover, if φ_t is a flow of a vector field $X \in \Gamma(TM)$ and $\mathcal{F}_t = \varphi_t(\mathcal{F})$, then

$$\frac{d}{dt}J^\top\varphi_t(x)_{t=t_0} = J^\top\varphi_{t_0}(x)\mathrm{div}_{\mathcal{F}_{t_0}}X, \tag{4.2}$$

where $J^\top\varphi_t$ is the Jacobian of φ_t restricted to leaves of $\mathcal{F}$. In particular,

$$\frac{d}{dt}J^\top\varphi_t(x)_{t=0} = \mathrm{div}_{\mathcal{F}}X.$$

Theorem 4.1. *Let $\mathcal{F}$ be a foliation on a Riemannian manifold (M,g). Let X be compactly supported vector field on M, which is admissible for p–modulus of $\mathcal{F}$. Assume there exists smooth extremal function f_0 for p–modulus of $\mathcal{F}$. Then, the following formula holds*

$$\frac{d}{dt}\mathrm{mod}_p(\mathcal{F}_t)^p_{t=0} = -p\int_M f_0^{p-1}\left(g(\nabla f_0, X) + f_0\mathrm{div}_{\mathcal{F}}\mathrm{X}\right)d\mu_M. \tag{4.3}$$

Before we prove above theorem we will need the following technical lemma.

Lemma 4.2. *Let $h : M\times\mathbb{R}^2 \to \mathbb{R}$ be defined as follows $h(x,t,s) = (f_s \circ \varphi_t)(x)$. Then*

$$\frac{\partial h}{\partial x}(x,t,s) = g(\nabla f_s(\varphi_t(x)), X_{\varphi_t(x)}), \quad \frac{\partial h}{\partial s}(x,t,s) = \frac{df_s}{ds}(\varphi_t(x)).$$

In particular,

$$\frac{d}{dt}(f_t\circ\varphi_t)(x) = g(\nabla f_t(\varphi_t(x)), X_{\varphi_t(x)}) + \frac{df_t}{dt}(\varphi_t(x)). \tag{4.4}$$

Proof. Consider two maps $F : M\times\mathbb{R}\to\mathbb{R}$ and $\varphi : M\times\mathbb{R}\to\mathbb{R}$,

$$F(x,s) = f_s(x), \quad \varphi(x,t) = \varphi_t(x).$$

Then $h = F\circ(\varphi, \mathrm{id}_{\mathbb{R}})$. Hence

$$\frac{\partial h}{\partial t} = h_*\frac{d}{dt} = F_*(\varphi_*\frac{d}{dt}, 0) = F_*(X,0) = f_{s*}X = g(\nabla f_s, X)$$

and

$$\frac{\partial h}{\partial t} = h_* \frac{d}{ds} = F_*(0, \frac{d}{ds}) = \frac{df_s}{ds}.$$

$\square$

Proof of Theorem 4.1. Since X is admissible for p–modulus of $\mathcal{F}$, then there are functions $h_1, h_2 \in L^p(M)$ such that conditions (A1) and (A2) hold. By Lemma 4.2 and (4.2) we have

$$\begin{aligned} |\frac{d}{dt}\left((f_t \circ \varphi_t)\, J^\top \varphi_t\right)| &= |\frac{d}{dt}\,(f_t \circ \varphi_t)\, J^\top \varphi_t + (f_t \circ \varphi_t)\, J^\top \varphi_t \mathrm{div}_{\mathcal{F}} \mathrm{X}| \\ &\leq |\frac{d}{dt}\,(f_t \circ \varphi_t)\,||J^\top \varphi_t| + |f_t \circ \varphi_t||J^\top \varphi_t||\mathrm{div}_{\mathcal{F}} \mathrm{X}| \\ &\leq C_1 h_2 + C_1 C_2 h_1, \end{aligned}$$

where $J^\top \varphi_t < C_1$, $t \in I$, and $\mathrm{div}_{\mathcal{F}} X < C_2$. Function $h = C_1 h_2 + C_1 C_2 h_1$ is in $L^p(M)$, hence, the existence of extremal function f_0 implies that $h \in L^1(L)$ for almost every leaf $L \in \mathcal{F}$ (Proposition 2.3). By Lebesgue dominated convergence theorem, Lemma 4.2 and (4.2) for any $L \in \mathcal{F}$ we have

$$\begin{aligned} \frac{d}{dt}\left(\int_L (f_t \circ \varphi_t) J^\top \varphi_t \, d\mu_L\right)_{t=0} &= \int_L \frac{d}{dt}((f_t \circ \varphi_t) J^\top \varphi_t)_{t=0}\, d\mu_L \\ &= \int_L (g(\nabla f_0, X) + (\frac{df_t}{dt})_{t=0} + f_0 \mathrm{div}_{\mathcal{F}} X)\, d\mu_L. \end{aligned}$$

Let L_t denotes the leaf of $\mathcal{F}_t$ i.e. $L_t = \varphi_t(L)$, $L \in \mathcal{F}$. Since, by Proposition 2.3, $\int_{L_t} f_t \, d\mu_{L_t} = 1$, then

$$0 = \frac{d}{dt}\left(\int_{L_t} f_t \, d\mu_{L_t}\right)_{t=0} = \frac{d}{dt}\left(\int_L (f_t \circ \varphi_t) J^\top \varphi_t \, d\mu_L\right)_{t=0}.$$

Hence, by above,

$$\int_L (\frac{df_t}{dt})_{t=0}\, d\mu_L = -\int_L (g(\nabla f_0, X) + f_0 \mathrm{div}_{\mathcal{F}} X)\, d\mu_L.$$

Notice that

$$\begin{aligned} \int_L (\frac{df_t}{dt})_{t=0}\, d\mu_L &= \int_L \frac{d}{dt}(f_t \circ \varphi_t \circ \varphi_t^{-1})_{t=0}\, d\mu_L \\ &= \int_L \frac{d}{dt}(f_t \circ \varphi_t) \frac{d}{dt}(\varphi_t^{-1})_{t=0}\, d\mu_L \\ &\leq C \int_L h_1 \, d\mu_L \end{aligned}$$

for some constant $C > 0$. Since $\mathrm{esssup}|\widehat{h_1}| < \infty$, it follows that

$$\mathrm{esssup}|\widehat{(\frac{df_t}{dt})_{t=0}}| < \infty.$$

Hence we may use Theorem 2.4 for $\varphi = (\frac{df_t}{dt})_{t=0}$. Moreover, we have

$$\int_M f_t^p \, d\mu_M = \int_{\varphi_t(M)} f_t^p \, d\mu_M = \int_M (f_t \circ \varphi_t)^p J\varphi_t \, d\mu_M$$

and

$$\begin{aligned} |\frac{d}{dt}((f_t \circ \varphi_t)^p J\varphi_t)| &= |p(f_t \circ \varphi_t)^{p-1} \frac{d}{dt}(f_t \circ \varphi_t) J\varphi_t + (f_t \circ \varphi_t)^p \frac{d}{dt}(J\varphi_t)| \\ &\leq C' h_1^{p-1} h_2 + C'' h_1^p \end{aligned}$$

for some constants $C', C'' > 0$. By Hölder inequality function $h_1^{p-1}h_2$ is integrable on M, hence function $C'h_1^{p-1}h_2 + C''h_1^p$ is integrable on M. Thus, by Lebesgue dominated convergence theorem and Lemma 4.2

$$\begin{aligned} \frac{d}{dt}\mathrm{mod}_p^p(\mathcal{F}_t)_{t=0} &= \frac{d}{dt}\left(\int_M f_t^p \, d\mu_M\right)_{t=0} \\ &= \frac{d}{dt}\left(\int_M (f_t \circ \varphi_t)^p J\varphi_t \, d\mu_L\right)_{t=0} \\ &= \int_M \frac{d}{dt}((f_t \circ \varphi_t)^p J\varphi_t)_{t=0} \, d\mu_M \\ &= \int_M (pf_0^{p-1} \frac{d}{dt}(f_t \circ \varphi_t)_{t=0} + f_0^p \mathrm{div} X) \, d\mu_M \\ &= \int_M (pf_0^{p-1}(g(\nabla f_0, X) + (\frac{df_t}{dt})_{t=0}) + f_0^p \mathrm{div} X) \, d\mu_M \\ &= \int_M (pf_0^{p-1} g(\nabla f_0, X) + f_0^p \mathrm{div} X) \, d\mu_M + p\int_M f_0^{p-1}(\frac{df_t}{dt})_{t=0} \, d\mu_M \\ &= \int_M \mathrm{div}(f_0^p X) + p\int_M f_0^{p-1}(\frac{df_t}{dt})_{t=0} \, d\mu_M \\ &= p\int_M f_0^{p-1}(\frac{df_t}{dt})_{t=0} \, d\mu_M. \end{aligned}$$

Finally, by Theorem 2.4, we obtain

$$\begin{aligned} \frac{d}{dt}\mathrm{mod}_p^p(\mathcal{F}_t)_{t=0} &= p\int_M f_0^p \widehat{(\frac{df_t}{dt})_{t=0}} \, d\mu_M \\ &= -p\int_M f_0^p (\widehat{g(\nabla f_0, X) + f_0}\mathrm{div}_{\mathcal{F}}\mathrm{X}) \, d\mu_M \\ &= -p\int_M f_0^{p-1}(g(\nabla f_0, X) + f_0 \mathrm{div}_{\mathcal{F}} X) \, d\mu_M. \end{aligned}$$

□

Variation of modulus implies the condition for tangent gradient of an extremal function (compare Corollary 4.4 [3])

Corollary 4.3. *Let $\mathcal{F}$ be a foliation on a Riemannian manifold (M, g). Assume all compactly supported vector fields X tangent to $\mathcal{F}$ are admissible*

for p–modulus of $\mathcal{F}$. Then

$$\nabla^{\top}(\log f_0) = \frac{1}{p-1} H_{\mathcal{F}^{\perp}}.$$

Proof. Let $X \in \Gamma(T\mathcal{F})$ be compactly supported. Then the flow φ_t of X maps $\mathcal{F}$ to $\mathcal{F}$, hence $\mathcal{F}_t = \mathcal{F}$ for all t. Thus, by Theorem 4.1 and formula 4.1, we have

$$\begin{aligned}
0 &= -p \int_M f_0^{p-1}(g(\nabla f_0, X) + f_0 \mathrm{div}_{\mathcal{F}} X)\, d\mu_M \\
&= -p \int_M f_0^{p-1}(g(\nabla f_0, X) + f_0(\mathrm{div}_M X + g(H_{\mathcal{F}^{\perp}}, X)))\, d\mu_M \\
&= \int_M (-p f_0^{p-1} g(\nabla f_0, X) - p f_0^p(\mathrm{div}_M X + g(H_{\mathcal{F}^{\perp}}, X)))\, d\mu_M \\
&= \int_M (-g(\nabla f_0^p, X) - p(\mathrm{div}_M(f_0^p X) - g(\nabla f_0^p, X)) - p f_0^p g(H_{\mathcal{F}^{\perp}}, X))\, d\mu_M \\
&= \int_M (g(p\nabla f_0^p, X) - g(\nabla f_0^p, X) - p f_0^p g(H_{\mathcal{F}^{\perp}}, X))\, d\mu_M.
\end{aligned}$$

Therefore, for compactly supported vector field $X \in \Gamma(T\mathcal{F})$

$$0 = \frac{d}{dt} \mathrm{mod}_p(\mathcal{F}_t)^p_{t=0} = p \int_M f_0^{p-1} g((p-1)\nabla f_0 - f_0 H_{\mathcal{F}^{\perp}}, X)\, d\mu_M.$$

Since $X \in \Gamma(T\mathcal{F})$ is arbitrary, it follows that

$$(p-1)\nabla^{\top} f_0 - f_0 H_{\mathcal{F}^{\perp}} = 0.$$

□

Let $\mathcal{F}$ be a p–admissible foliation on a Riemannian manifold M. We say that $\mathcal{F}$ is a *critical point* of a functional

$$\mathcal{F} \mapsto \mathrm{mod}_p(\mathcal{F}), \tag{4.5}$$

if for any compactly supported vector field X we have $\frac{d}{dt}\mathrm{mod}_p(\mathcal{F}_t) = 0$, where $\mathcal{F}_t = \varphi_t(\mathcal{F})$ and φ_t is a flow of X.

The following result gives the characterization of critical points.

Corollary 4.4. *Let $\mathcal{F}$ be a p–admissible foliation on a Riemannian manifold M. Then, $\mathcal{F}$ is a critical point of* (4.5) *if and only if*

$$\nabla(\log f_0^p) = pH_{\mathcal{F}} + qH_{\mathcal{F}^{\perp}}. \tag{4.6}$$

where f_0 is an extremal function for p–modulus of $\mathcal{F}$ and p, q are conjugate coefficients i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We begin proof by stating general facts. By Corollary 4.3

$$\nabla^{\top}(\log f_0^p) = \frac{p}{p-1} H_{\mathcal{F}^{\perp}} = qH_{\mathcal{F}^{\perp}}. \tag{4.7}$$

Moreover, since $\mathrm{div}_{\mathcal{F}} X = g(X, H_{\mathcal{F}})$ for $X \in \Gamma(T^{\perp}\mathcal{F})$, then

$$\frac{d}{dt}\mathrm{mod}_p(\mathcal{F}_t)^p_{t=0} = -p \int_M g(\nabla f_0^p - f_0^p H_{\mathcal{F}}, X)\, d\mu_M \tag{4.8}$$

for all compactly supported vector fields $X \in \Gamma(T^{\perp}\mathcal{F})$.

Assume $\mathcal{F}$ is a critical point of a functional (4.5). Then by (4.8) we get $\nabla^{\perp} f_0^p = f_0^p H_{\mathcal{F}}$, hence

$$\nabla^{\perp}(\log f_0^p) = pH_{\mathcal{F}}.$$

This, together with (4.7), implies (4.6).

Assume now (4.6) holds. Right–hand side of (4.3) is linear with respect to X. Moreover, for X tangent to $\mathcal{F}$ we have (compare proof of Corollary 4.3) $\frac{d}{dt}\mathrm{mod}_p(\mathcal{F}_t)_{t=0} = 0$. Hence, by (4.8), it suffices to show that

$$\int_M g(\nabla f_0^p - f_0^p H_{\mathcal{F}}, X)\, d\mu_M = 0$$

for all compactly supported vector fields $X \in \Gamma(T^{\perp}\mathcal{F})$. This follows by assumption (4.6), which implies $\nabla^{\perp}(f_0^p) = f_0^p H_{\mathcal{F}}$. □

Now, we consider the case of two orthogonal foliations i.e. we assume that for a given foliation $\mathcal{F}$ on a Riemannian manifold (M, g) the distribution $\mathcal{G} = \mathcal{F}^{\perp}$ is integrable. Let $p, q > 1$, be conjugate coefficients i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 4.5. *Assume $\mathcal{F}$ is p–admissible and $\mathcal{G}$ is q–admissible. Then the following conditions are equivalent:*

(1) *$\mathcal{F}$ and $\mathcal{G}$ are a critical points of the functionals*

$$\mathcal{F} \mapsto \mathrm{mod}_p(\mathcal{F}) \quad \textit{and} \quad \mathcal{G} \mapsto \mathrm{mod}_q(\mathcal{G}),$$

respectively,

(2) *the pair $(\mathcal{F}, \mathcal{G})$ is a critical point of a functional*

$$(\mathcal{F}, \mathcal{G}) \mapsto \mathrm{mod}_p(\mathcal{F})\mathrm{mod}_q(\mathcal{G}),$$

(3) *the extremal functions f_0 of p–modulus of $\mathcal{F}$ and g_0 of q–modulus of $\mathcal{G}$ are related as follows*

$$\mathrm{mod}_q(\mathcal{G})^q \cdot f_0^p = \mathrm{mod}_p(\mathcal{F})^p \cdot g_0^q. \tag{4.9}$$

Proof. (1) ⇒ (2) Follows from the equality

$$\frac{d}{dt}(\mathrm{mod}_p(\mathcal{F}_t)\mathrm{mod}_q(\mathcal{G}_t))_{t=0} = \tag{4.10}$$

$$\frac{d}{dt}\mathrm{mod}_p(\mathcal{F}_t)_{t=0} \cdot \mathrm{mod}_q(\mathcal{G}) + \mathrm{mod}_p(\mathcal{F}) \cdot \frac{d}{dt}\mathrm{mod}_q(\mathcal{G}_t)_{t=0}.$$

(2) ⇒ (1) By existence of extremal functions of p–modulus of $\mathcal{F}$ and q–modulus of $\mathcal{G}$, it follow by Proposition 2.1 that p–modulus of $\mathcal{F}$ and q–modulus of $\mathcal{G}$ are positive. If $X \in \Gamma(T^{\perp}\mathcal{F})$, then $\frac{d}{dt}\mathrm{mod}_q(\mathcal{G}_t)_{t=0} = 0$, hence,

by (4.10),

$$\frac{d}{dt}\mathrm{mod}_p(\mathcal{F}_t)_{t=0} = 0.$$

If $X \in \Gamma(T\mathcal{F})$, then $\mathrm{mod}_p(\mathcal{F}_t)_{t=0} = 0$. Since the variation of modulus is linear with respect to X, it follows that $\frac{d}{dt}\mathrm{mod}_p(\mathcal{F}_t)_{t=0} = 0$ for any compactly supported vector field X. Analogously $\frac{d}{dt}\mathrm{mod}_q(\mathcal{G}_t)_{t=0} = 0$ for any compactly supported vector field X.

(1) $\Leftrightarrow$ (3) By Corollary 4.4 condition (1) is equivalent to the following

$$\nabla(\log f_0^p) = pH_{\mathcal{F}} + qH_{\mathcal{G}} = \nabla(\log g_0^q). \tag{4.11}$$

Assume (1) holds. Then by (4.11), $f_0^p = Cg_0^q$ for some constant $C > 0$. Hence f_0 and g_0 are Hölder dependent. Thus

$$\begin{aligned}
\mathrm{mod}_p(\mathcal{F})\mathrm{mod}_q(\mathcal{G}) &= \left(\int_M f_0^p d\mu_M\right)^{\frac{1}{p}} \left(\int_M g_0^q\, d\mu_M\right)^{\frac{1}{q}} = \int_M f_0 g_0\, d\mu_M \\
&= \int_M C^{\frac{1}{p}} g_0^{\frac{q}{p}+1}\, d\mu_M = C^{\frac{1}{p}} \int_M g_0^q\, d\mu_M \\
&= C^{\frac{1}{p}} \mathrm{mod}_q(\mathcal{G}_0)^q.
\end{aligned}$$

Therefore

$$C = \frac{\mathrm{mod}_p(\mathcal{F})^p}{\mathrm{mod}_q(\mathcal{G})^q},$$

so (4.9) holds.

Assume now f_0 and g_0 are Hölder dependent and (4.9) holds. Thus

$$\nabla(\log f_0^p) = \nabla(\log g_0^q). \tag{4.12}$$

By Corollary 4.3 we have

$$\nabla^{\mathcal{F}}(\log f_0^p) = qH_{\mathcal{G}} \quad \text{and} \quad \nabla^{\mathcal{G}}(\log g_0^q) = pH_{\mathcal{F}},$$

where $\nabla^{\mathcal{F}}$ and $\nabla^{\mathcal{G}}$ denote tangent to $\mathcal{F}$ and to $\mathcal{G}$ part of the gradient, respectively. By above equalities and by (4.12)

$$\nabla(\log f_0^p) = \nabla^{\mathcal{F}}(\log f_0^p) + \nabla^{\mathcal{G}}(\log g_0^q) = qH_{\mathcal{G}} + pH_{\mathcal{F}},$$

hence, by Corollary 4.4, $\mathcal{F}$ is a critical point of a functional (4.5). Analogously, $\mathcal{G}$ is a critical point of a functional (4.5) with a coefficient q. Therefore (1) holds. $\square$

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References

[1] L. Ahlfors, A. Beurling, *Conformal invariants and function–theoretic null–sets*, Acta Math. 83 (1950), 101–129.

[2] D. Blachowska, *Some properties of a modulus of foliation*, Bull. Soc. Sci. Lett. Lodz Ser. Rech. Deform. 41 (2003), 5–11.

[3] M. Ciska, *On extremal function of a foliation*, Preprint.

[4] B. Fuglede, *Extremal length and functional completion*, Acta Math. 98 (1957) 171–219.

[5] J. Kalina, A. Pierzchalski, *A variation of the modulus of submanifold families*, Analytic functions, Kozubnik 1979 (Proc. Seventh Conf., Kozubnik, 1979), pp. 250–257, Lecture Notes in Math., 798, Springer, Berlin, 1980.

Department of Mathematics and Computer Science
University of Łódź
ul. Banacha 22, 90-238 Łódź
Poland
E-mail address: mciska@math.uni.lodz.pl